Graph Theory
Homework 8

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Proposition 0.1 (Exercise 1). *Let $G = (V, E)$ be a connected graph with $|V(G)| = n$ and maximum degree Δ and Laplacian L. Let $0 = \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$ be the eigenvalues of L. Then $\lambda_n \leq 2\Delta$, with equality if and only if G is bipartite and regular.*

Proof. Let $f \in \mathbb{R}^n$ be an eigenvector of L with eigenvalue λ_n. Choose $v_0 \in V(G)$ so that $|f(v_0)| \geq |f(v_i)|$ for all i. By scaling, we may assume that $|f(v_0)| = 1$. Then

$$|\lambda_n| = |\lambda_n f(v_0)| = |(Lf)(v_0)| = \left|(\deg v_0)f(v_0) - \sum_{w \in \Gamma(v_0)} f(w)\right|$$
$$\leq \deg v_0 + \left|\sum_{w \in \Gamma(v_0)} f(w)\right| \leq \deg v_0 + \sum_{w \in \Gamma(v_0)} |f(w)| \leq \Delta + \sum_{i=1}^{\Delta} |1| = 2\Delta$$

If we have equality, then $\deg v_0 = \Delta$, and $f(w) = -1$ for $w \in \Gamma(v_0)$. Then all neighbors of v_0 also have $|f(w)|$ maximal, so the same chain of inequalities shows that $\deg w = \Delta$, and $f(v) = 1$ for $v \in \Gamma(w)$. Since G is connected, this process inductively propagates to all vertices of G, so G is regular and bipartite (one color for +1 and another color for -1 vertices.)

Conversely, suppose that G is bipartite and regular. Then $f \in \mathbb{R}^n$ which takes value 1 on one maximal independent set and value -1 on the remaining vertices is an eigenvector of L with eigenvalue Δ. □

Proposition 0.2 (Exercise 2a). *Let A_n be the adjacency matrix of the cycle graph C_n and L_n be the Laplacian of C_n. The spectra $\{\mu_i\}$ of A_n and $\{\lambda_i\}$ of L_n are*

$$\{\mu_i\} = \left\{\zeta + \zeta^{-1} : \zeta^n = 1\right\}$$
$$\{\lambda_i\} = \left\{2 - \zeta - \zeta^{-1} : \zeta^n = 1\right\}$$

(Note that these always turn out to be real.)

Proof. The adjacency matrix of C_n is

$$A_n = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 1 \\ 1 & 0 & 1 & 0 & \dots & \\ 0 & 1 & 0 & 1 & \dots & \\ 0 & 0 & 1 & 0 & \ddots & \\ \vdots & \vdots & \vdots & \vdots & \ddots & 1 \\ 1 & & & & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & \\ 0 & 0 & 0 & 1 & \dots & \\ 0 & 0 & 0 & 0 & \ddots & \\ \vdots & \vdots & \vdots & \vdots & \ddots & 1 \\ 1 & & & & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & \\ 0 & 0 & 0 & 1 & \dots & \\ 0 & 0 & 0 & 0 & \ddots & \\ \vdots & \vdots & \vdots & \vdots & \ddots & 1 \\ 1 & & & & 0 & 0 \end{pmatrix}^T$$

denote the matrices on the RHS by B_n and B_n^T. Using the method of expansion by cofactors along the first column, we can compute that the characteristic polynomial of B_n is $\lambda^n - 1$. That is, the eigenvalues of B_n are precisely the nth roots of unity. The eigenvector of B_n associated to an nth root of unity ζ is $v_\zeta = (1, \zeta, \zeta^2, \dots, \zeta^{n-1})$. Similarly, $B_n v_\zeta = \zeta^{-1} v_\zeta$. Thus

$$A_n v_\zeta = B_n v_\zeta + B_n^T v_\zeta = \zeta v_\zeta + \zeta^{-1} v_\zeta = (\zeta + \zeta^{-1}) v_\zeta$$

Since there are n such v_ζ, this gives all the eigenvectors and eigenvalues of A_n. Now let L_n be the Laplacian of C_n. Since C_n is 2-regular, $L_n = 2I_n - A_n$. Thus L_n has the same eigenvectors v_ζ as A_n, with eigenvalues

$$L_n v_\zeta = 2I_n v_\zeta - A_n v_\zeta = 2v_\zeta - (\zeta + \zeta^{-1}) v_\zeta = (2 - \zeta - \zeta^{-1}) v_\zeta$$

□

Let's take a moment to verify that the results of Theorem 5 (page 263 of Bollobas) hold for the spectra above.

1. Since $|\zeta| = 1$, we have $|\mu| = |\zeta + \zeta^{-1}| \leq |\zeta| + |\zeta^{-1}| \leq 2$.

2. The maximal degree Δ for C_n is 2, and C_n is 2-regular, so Δ should be an eigenvalue of A_n. And it is, since $\zeta = 1$ is always an nth root of unity, and $1 + 1^{-1} = 2$.

3. When n is even, C_n is bipartite, in which case -1 is an nth root of unity and $-1 + -1^{-1} = -2$ is an eigenvalue of A_n.

4. When n is even, C_n is biparite, so the spectrum should be symmetric about zero. If ζ is an nth root of unity and n is even, then $-\zeta$ is also an nth root of unity, so if $\mu = \zeta + \zeta^{-1}$ is an eigenvalue, then $-\zeta + (-\zeta^{-1}) = -(\zeta + \zeta^{-1}) = \mu$ is also an eigenvalue.

5. Also note that the result from Exercise 1 is satisfied, since $\zeta = -1$ is an eigenvalue of the Laplacian precisely when n is odd, and $2 - (-1) - (-1)^{-1} = 4 = 2\Delta$.

Proposition 0.3 (Exercise 2b)**.** *Let $A_{n,m}$ be the adjacency matrix of the complete bipartite graph $K_{n,m}$, and let $L_{n,m}$ be the Laplacian. The spectra $\{\mu_i\}$ of $A_{n,m}$ is*

$$\{\mu_i\} = \{0, \pm\sqrt{mn}\}$$

where $\sqrt{mn}, -\sqrt{mn}$ have multiplicity one and zero has multiplicity $m+n-2$. The spectrum $\{\lambda_i\}$ of $L_{n,m}$ is

$$\{\lambda_i\} = \{0, m, n, m+n\}$$

Proof. If we order the vertices of $K_{n,m}$ so that the first n vertices are one independent set and the other m vertices come after, then the matrix $A_{n,m}$ has a very simple block form.

$$A_{n,m} = \begin{pmatrix} 0 & \dots & 0 & 1 & \dots & 1 \\ \vdots & & \vdots & \vdots & \vdots & \\ 0 & \dots & 0 & 1 & \dots & 1 \\ 1 & \dots & 1 & 0 & \dots & 0 \\ \vdots & & \vdots & \vdots & \vdots & \\ 1 & \dots & 1 & 0 & \dots & 0 \end{pmatrix}$$

Let $x = (x_1, \dots, x_n, x_{n+1}, \dots, x_{n+m})$, then

$$A_{n,m}x = \begin{pmatrix} \sum_{i=n+1}^{n+m} x_i \\ \vdots \\ \sum_{i=n+1}^{n+m} x_i \\ \sum_{i=1}^{n} x_i \\ \vdots \\ \sum_{i=1}^{n} x_i \end{pmatrix}$$

Thus if x is an eigenvector of $A_{n,m}$ with eigenvalue $\mu \neq 0$, then x has the form $(a, a, a, \dots, b, b, b, \dots)$, and then the eigenvalue must be $\pm\sqrt{mn}$ (mild computations omitted), provided $a \neq 0$ or $b \neq 0$. Thus the only nonzero eigenvalues of $A_{n,m}$ are $\pm\sqrt{mn}$. The associated eigenvectors are $(\sqrt{m}, \sqrt{m}, \dots, \sqrt{n}, \sqrt{n}, \dots)$ and $(-\sqrt{m}, -\sqrt{m}, \dots, \sqrt{n}, \sqrt{n}, \dots)$. As we computed, these span the eigenspace associated to $\mu = \sqrt{mn}$, so all the other eigenvalues are zero, so the eigenvalue zero has multiplicity $m + n - 2$.

Now consider $L_{n,m}$, which also has a simple form.

$$L_{n,m} = \begin{pmatrix} m & 0 & \dots & 0 & -1 & -1 & \dots & -1 \\ 0 & m & \dots & 0 & -1 & -1 & \dots & -1 \\ \vdots & & \ddots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & m & -1 & -1 & \dots & -1 \\ -1 & -1 & \dots & -1 & n & 0 & \dots & 0 \\ -1 & -1 & \dots & -1 & 0 & n & \dots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \ddots & \vdots \\ -1 & -1 & \dots & -1 & 0 & \dots & 0 & n \end{pmatrix}$$

Let $x = (x_1, \dots, x_{n+m})$, then

$$L_{n,m}x = \begin{pmatrix} mx_1 - \sum_{i=n+1}^{n+m} x_i \\ mx_2 - \sum_{i=n+1}^{n+m} x_i \\ \vdots \\ mx_n - \sum_{i=n+1}^{n+m} x_i \\ nx_{n+1} - \sum_{i=1}^{n} x_i \\ \vdots \\ nx_{n+m} - \sum_{i=1}^{n} x_i \end{pmatrix}$$

We know that 0 is an eigenvalue with eigenspace of dimension 1. We also see by inspection that $m+n$ is an eigenvalue, with eigenvector $(m, \dots, m, -n, \dots, -n)$. The equation $L_{m,n}x = (m+n)x$ constrains all the entries x_i, so the eigenvalue has multiplicity 1.

By looking at $L_{n,m}$, we guess that m, n should be eigenvalues as well, and they are. If $x = (x_1, \dots, x_{n+m})$ is an eigenvector with eigenvalue m, we get constraint equations

$$mx_j - \sum_{i=n+1}^{n+m} x_i = mx_j \qquad \text{for } j = 1, \dots, n$$
$$nx_j - \sum_{i=1}^{n} x_i = mx_j \qquad \text{for } j = n+1, \dots, n+m$$

The first equation says $\sum_{i=n+1}^{n+m} x_i = 0$, and the second implies $x_j = \left(\sum_{i=1}^{n} x_i\right)/(m-n)$ for $n+1 \le j \le n+m$, so $x_{n+1} = \dots = x_{n+m}$. But then $x_{n+1} = \dots = x_{n+m} = 0$, by the first equation. Then looking again at the second equation, $\sum_{i=1}^{n} x_i = 0$. These are the only constraints on the eigenspace associated to m, so we have the following basis for the m eigenspace.

$$e_1 - e_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, e_1 - e_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \\ \vdots \end{pmatrix}, \dots, e_1 - e_n = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ -1 \\ 0 \\ \vdots \end{pmatrix}$$

Thus the eigenvalue m has multiplicity $n-1$. Since we have complete symmetry between m and n, the eigenvalue n has multiplicity $m-1$. Thus

$$\{\lambda_i\} = \{0, m, n, m+n\}$$

and there can be no others because these eigenvalues counted with multiplicity give $n+m$ eigenvalues. □

We pause to check that the spectra computed in 2b satisfy Theorem 5 (page 263) of Bollobas.

1. $K_{n,m}$ is regular if and only if $n = m$, in which case $\sqrt{mn} = \sqrt{m^2} = m = \Delta$ is an eigenvalue.

2. If $-\Delta$ is an eigenvalue, then $-\Delta = -\sqrt{mn}$ which implies that mn is a square, so $m = n$ and $K_{m,n}$ is regular (it is always bipartite, of course).

3. $K_{n,m}$ is bipartite, and the spectrum of the adjacency matrix is symmetric.

4. Also note that the property from Exercise 1 holds, $\lambda_n = m + n \le 2\Delta = 2\max(m, n)$, with equality if and only if $m = n$, in which case $K_{m,n}$ is regular (and bipartite).